MDE and Model Validation

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Outline

Validation standards

- "The term model refers to a quantitative method, system, or approach that applies statistical, economic, financial, or mathematical theories, techniques, and assumptions to process input data into quantitative estimates." - Federal Reserve
- "Model validation is the set of processes and activities intended to verify that models are performing as expected, in line with their design objectives and business uses. Effective validation helps to ensure that models are sound, identifying potential limitations and assumptions and assessing their possible impact. All model components—inputs, processing, outputs, and reports—should be subject to validation; this applies equally to models developed in-house and to those purchased from or developed by vendors or consultants."

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Key steps in model validation

1. An evaluation of the soundness of underlying conceptual framework (e.g., economic, statistical/mathematical).
2. An evaluation of the soundness and quality of software implementation and the process used in building the model.
3. An evaluation of the appropriateness of use of a model.
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Main focus of research work

- The main focus of the initial research is the validation of models required to be specified mathematically e.g., pricing and risk models.
- We will be discussing some observations from application of step 2 above to validating the Binomial CRR model for pricing European stock options. This model will be compared to the Black-Scholes model as a basis for comparing the conceptual soundness of the Binomoal model.
- We sketch some desirable features of an MDE (principally for specification and proof).

Some key CRR assumptions

- There are no arbitrage opportunities.
- Interest rates are constant over the life of the option and can be considered as continuously compounded.
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European options

Definition

(European options). A European option provides the holder the right to buy (call) or sell (put) an underlying asset at a specified time, t, for a fixed price, K. A call option has a payout at time t of $max\left(S_t - K, 0\right)$ and the put $max\left(K - S_t, 0\right)$.

Arbitrage

- Assuming the current time is $t = 0$ and the stock price is $S_t = S_0$, the price at $t = 1$ will rise to $S_0 u$ with probability p or fall to $S_0 d$ with probability $1 - p$ where $u > 1, 0 < d < 1$. Hence at $t = 1$ we have

$$S_1 = \begin{cases} S_0 u & \textit{with probability } p \\ S_0 d & \textit{with probability } 1 - p \end{cases}$$

- To avoid arbitrage we require $d < e^r < u$ where r is the continuously compounded interest rate, typically assumed to be greater than zero. This guarantees that the probability p is positive and satisfies the following $0 < p < 1$.

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Replicating portfolio

- This consists of holding positions in assets (i.e., the stock and cash) which produce the same set of cash flows as a derivative. For the European option this implies

$$V_0 = h_1 S_0 + h_2. \tag{1}$$

- At the next timepoint Δt from now

$$V_{1(\equiv \Delta t)} = \begin{cases} h_1 S_0 u + h_2 e^{r\Delta t} = O_u & \textit{with probability } p \\ h_1 S_0 d + h_2 e^{r\Delta t} = O_d & \textit{with probability } 1 - p \end{cases}$$

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Replicating portfolio continued

- Solving for h_1, h_2 we get

$$h_1 = \frac{O_u - O_d}{S_0 (u - d)}$$
$$h_2 = e^{-r\Delta t} \left(\frac{O_d u - O_u d}{u - d} \right).$$

- Substituting into (1) we get

$$V_0 = e^{-r\Delta t} \left[O_u \left(\frac{e^{r\Delta t} - d}{u - d} \right) + O_d \left(1 - \frac{e^{r\Delta t} - d}{u - d} \right) \right]$$
$$V_0 = e^{-r\Delta t} \left[O_u p + O_d (1 - p) \right]$$

Multiperiod binomial

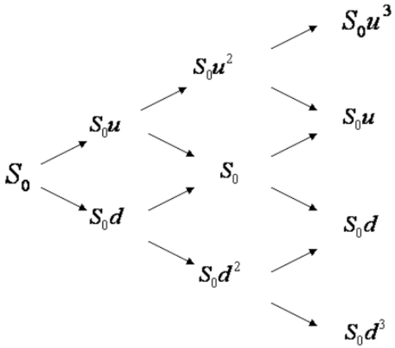

Figure: 3-period Binomial model

Valuation of multiperiod binomial model

- The valuation is:

$$C(S_0, T)_N = e^{-rT}\left[\sum_{n=0}^{N}\binom{N}{n}p^n(1-p)^{N-n}\max\left(0, u^n d^{N-n}S_0 - X\right)\right]$$

- Where $\binom{N}{n} = \frac{N!}{(N-n)!n!}$.
- The valuation is zero (0) for values of n where it is not the case that $u^n d^{N-n} S_0 - X > 0$. This implies that

$$n > \frac{\ln\left(\frac{X}{S_0 d^N}\right)}{\ln\left(\frac{u}{d}\right)}.$$

Valuation continued

- Let a denote the smallest positive integer (corresponding to the number of upward moves) such that the option finishes in the money, that is $u^a d^{N-a} S_0 > X$ then we can write

$$a = \frac{\ln(X/S_0) - N \ln(d)}{\ln(u/d)} + \varepsilon = n.$$

- The valuation formula becomes
$C(S_0, T)_N = S_0 B\left(n \geq a | N, p^{'}\right) - Xe^{-rT} B(n \geq a | N, p)$
where
- $p^{'} \equiv upe^{-r\Delta t}$ and $B(n \geq a | N, p) = \sum_{n=a}^{N} \binom{N}{n} p^n (1-p)^{N-n}$.

Deriving parameters

- By equating the mean and variance of binomial stock returns to that of geometric brownian motion stock returns we eventually get (after some substitutions and simplications)

$$e^{\mu\Delta_t}\left[\frac{1}{d}+d\right]-1-e^{2\mu\Delta_t}=\sigma^2\Delta_t.$$

- The above can be solved by noticing that it is quadratic in d. Using the quadratic formula and assuming that $e^Y \approx 1+Y$ for small Y and that terms involving Δ_t^2 vanish as Δ_t becomes smaller we get $d = e^{-\sigma\sqrt{\Delta t}}$. We can also infer u by assuming $u = 1/d$.

Implications for p

- We know that

$$p = \frac{e^{\mu \Delta_t} - d}{u - d}.$$

- To avoid arbitrage we require the numerator to be greater than 0, which implies

$$\sigma > r\Delta t.$$

- This result is produced after substituting for u, d in p and assuming that $e^{Y} \approx 1 + Y$ for small Y .
- This result has implications for the number of intervals N and ensures $p \in (0, 1)$, $p^{'} \in (0, 1)$.

Convergence

- Given the following

$$\frac{S_{t+\Delta_t}}{S_t} = e^{\left(\mu - \sigma^2/2\right)\Delta_t + \sigma\left(W_{t+\Delta_t} - W_t\right)}.$$

- Expanding the above exponential up to powers of two but ignoring terms greater than Δ_t we get

$$1 + \left(\mu - \sigma^2/2\right)\Delta_t + \sigma Z\sqrt{\Delta_t} + \frac{\sigma^2}{2}\Delta_t Z^2.$$

- Where $Z \sim N(0,1)$. The expectation and variance of the expression $(S_{t+\Delta t}/S_t) - 1$ (the percentage return) is $\mu\Delta_t$ and $\sigma^2\Delta_t$.
- It can be shown that $(S_{t+N\Delta t}/S_t) - 1$ has mean and variance μT and $\sigma^2 T$.

Convergence continued

- Let $n \sim \mathcal{B}(N, p)$ be a binomial distributed random variable denoting the number of up moves then the stock price at maturity $S_N^* \equiv S_T$ as a ratio to an initial price S_0 equals $S_N^*/S_0 = u^n d^{N-n}$.
- The limit as $N \to \infty$ of

$$\begin{aligned}
\mathrm{E}\left[\ln\left(S_N^*/S_0\right)\right] =& \mathrm{E}\left[n \ln\left(u/d\right) + N \ln d\right] \\
=& \mu T \\
\mathrm{Var}\left[\ln\left(S_N^*/S_0\right)\right] =& \mathrm{Var}\left[n \ln\left(u/d\right) + N \ln d\right] \\
=& \sigma^2 T.
\end{aligned}$$

Convergence to Black-Scholes

- It is well-known that that the binomial distribution can be approximated by that of the normal distribution where $N \to \infty$ in the following

$$f_N(n) = \binom{N}{n} p^n (1-p)^{N-n}.$$

- To determine the limits of the integration in this normal distribution the value a (defined earlier) must be transformed into a number corresponding to a limit of a standard normal distribution.

$$d = -\frac{(a - \mathrm{E}(n))}{\sqrt{\mathrm{Var}(n)}}.$$

Convergence to Black-Scholes continued

- Substituting the expressions for $\mathrm{E}(n), \mathrm{Var}(n)$ in d we get

$$d = \frac{\ln(S_0/X) + \mathrm{E}(\ln(S_T/S_0))}{\sqrt{\mathrm{Var}(\ln(S_T/S_0))}} - \frac{\varepsilon \ln(u/d)}{\sqrt{\mathrm{Var}(\ln(S_T/S_0))}}.$$

- After a simplification and a number of other assumptions, it can be shown that

$$d = \frac{\ln(S_0/X) + \left(r - \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}} = d_2.$$

- Moreover, it can be shown that $d_1 = d_2 + \sigma\sqrt{T}$.

Key considerations

- Demonstrating conceptual soundness, if done properly, can require several non-trivial proofs
- Such proofs can give rise to constraints on model parameters which should also be incorporated into model implementation
- Undertaking proofs requires making assumptions, these should also be checked/validated
- A suitable MDE should aid users in the formulation and derivation of proofs and subsequent auto-translation to code

Where MDE could help - specifications

- Solving for h_1, h_2 could be aided by having formalisms handling statements like: $\textbf{Solve}\ V_1\ \textbf{for}\ h_1, h_2\ \textbf{Sub in}\ V_0$
- Proving $0 \leq \varepsilon < 1$ could be achieved by having statements like

$$
\begin{aligned}
v : \mathbb{R} &= \ln\left(X/S_0 d^N\right) / \ln\left(u/d\right) \\
a : \mathbb{I} &\triangleq \min\left\{x : x \in \mathbb{I}, x > v\right\} \\
\varepsilon &: \mathbb{R} \\
a &= v + \varepsilon \\
\textbf{Prove}\ \varepsilon &: [0, 1)
\end{aligned}
$$

Where MDE could help - specifications continued

- Features for simplifying expressions could help in deriving parameters e.g., $e^{\mu\Delta_t}\left[\frac{1}{d}+d\right]-1-e^{2\mu\Delta_t}=\sigma^2\Delta_t$ could be solved for d. An example could be

$$\textbf{Simplify} e^{\mu\Delta_t}\left[\frac{1}{d}+d\right]-1-e^{2\mu\Delta_t}=\sigma^2\Delta_t \textbf{ as Quadratic in } d$$

$$\textbf{by Expanding } \mathbf{e^{Y}} \textbf{ to First}$$

$$\textbf{by Ignoring } \Delta_t \textbf{ Terms Greater } 1$$

- Other examples would include the simplication of the product of expressions such as that of the product of log of returns along with the expectation and variance of this value.

Conditions for implementation

- Statements like preconditions and asserts could be used to provide checks in implementation of specifications. A stylised version

$$\textbf{Implement}\ S_0 B\left(n \geq a|N, p'\right) - Xe^{-rT} B\left(n \geq a|N, p\right)$$

$$\textbf{Preconditions}\left[\sigma > r\Delta_t\right]$$

$$\textbf{Assert}\{$$

$$v : \mathbb{R} = \ln\left(X/S_0 d^N\right)/\ln\left(u/d\right)$$

$$a : \mathbb{I} \triangleq \min\left\{x : x \in \mathbb{I}, x > v\right\}$$

$$a = v + \varepsilon$$

$$\textbf{Where}\ \varepsilon : [0, 1)\}$$